

# THE HOMOLOGY OF CONNECTIVE MORAVA $E$ -THEORY WITH COEFFICIENTS IN $\mathbb{F}_p$

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ABSTRACT. Let  $e_n$  be the connective cover of the Morava  $E$ -theory spectrum  $E_n$  of height  $n$ . In this paper we compute its homology  $H_*(e_n; \mathbb{F}_p)$  for any prime  $p$  and  $n \leq 4$  up to possible multiplicative extensions when  $n$  is 3 or 4. We do this by using the Künneth spectral sequence based on  $BP$  which we prove is multiplicative.

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## 1. INTRODUCTION

In this paper we compute  $H_*(e_n; \mathbb{F}_p)$  where  $e_n$  is the connective cover of height  $n$  Morava  $E$ -theory at the prime  $p$  when  $n \leq 4$ , see [Rez98] for details on  $E_n$ . While periodic Morava  $E$ -theory,  $E_n$ , has no homology with coefficients in  $\mathbb{F}_p$ , we find that  $H_*(e_n; \mathbb{F}_p)$  contains  $H(BP\langle n \rangle; \mathbb{F}_{p^n})$  in our range.

We hope that this computation will help in understanding certain chromatic phenomena. At the core of the computation is the ring  $\pi_*(H\mathbb{F}_p \wedge_{BP} e_n)$ . We think that the new classes  $f_I \in \pi_*(H\mathbb{F}_p \wedge_{BP} e_n)$  may be of interest in light of the recent work of Tyler Lawson in which he establishes that  $BP\langle n \rangle$  can not be  $E_\infty$  ring spectra when  $p = 2$  and  $n \geq 4$ , see [Law17]. As it is known that  $e_n$  does admit such an  $E_\infty$  structure, it seems plausible that the classes in  $H_*(e_n; \mathbb{F}_p)$  coming from  $\pi_* H\mathbb{F}_p \wedge_{BP\langle n \rangle} e_n$  will account for this discrepancy.

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Our method of computation is relatively straightforward. We use the Künneth spectral sequence based on  $BP$  and show that it collapses. We show this collapse by using a result of Baker and Richter from [BR08]. In order to apply this result we must show that the relevant Künneth spectral sequence, denoted KSS throughout, is multiplicative. We base our spectral sequence on  $BP$  which is known to be an  $E_4$ -algebra by remarkable work of Basterra and Mandell, see [BM13]. We are able to show that

$$\mathrm{Tor}_s^{\pi_* BP}(\mathrm{H}_*(BP; \mathbb{F}_p), \pi_* e_n)_t \Rightarrow \mathrm{H}_{s+t}(e_n; \mathbb{F}_p)$$

is multiplicative in Section 2. This crucially uses work of Mandell from [Man12] on categories of modules over  $E_4$ -algebras. We then arrive at our theorem. We use the notation  $[n] := \{1, \dots, n\}$ ,  $w(i) := p^i - 1$  and  $m(A, B) := \#\{(a, b) \in A \times B \mid a > b\}$ .

**Theorem 1.1.** *When  $n \leq 4$ , the Künneth spectral sequence converging to the homology of the connective Morava  $E$ -theory spectrum  $e_n$  with coefficients in  $\mathbb{F}_p$  collapses at the  $E_2$ -page. Thus we have an isomorphism of the  $E_\infty$ -page of the spectral sequence*

$$E^0(\mathrm{H}_*(e_n; \mathbb{F}_p)) \cong \mathrm{H}_*(BP; \mathbb{F}_{p^n}) \otimes_{\mathbb{F}_{p^n}} A/\mathfrak{a} \otimes_{\mathbb{F}_{p^n}} E_{\mathbb{F}_{p^n}}[\bar{p}, \bar{v}_{n+1}, \bar{v}_{n+2}, \dots].$$

where  $E_{\mathbb{F}_{p^n}}[\dots]$  denotes an exterior algebra over  $\mathbb{F}_{p^n}$  with the indicated generators.  $A$  is the exterior algebra  $E_{e_n*}[f_I \mid I \subseteq [n]]$  and  $\mathfrak{a}$  is the ideal generated by the following relations:

$$\begin{aligned} & u^{w(i)} u_i \quad \text{for } 0 \leq i \leq n; \\ & u^{w(\min I)} f_I \quad \text{for } I \subseteq [n]; \\ & u_a f_{I \cup b} - u_b f_{I \cup a} \quad \text{for } I \subseteq [n], \ a, b \in [n] \text{ with } a, b < \min(I). \end{aligned}$$

and

$$f_I \cdot f_J - \begin{cases} (-1)^{m(I \setminus i_0, J)} u_{i_0} f_{(I \setminus i_0) \cup J} & \text{if } i_0 \geq j_0 \text{ and } (I \setminus i_0) \cap J = \emptyset; \\ (-1)^{m(I, J \setminus j_0)} u_{j_0} f_{I \cup (J \setminus j_0)} & \text{if } j_0 \geq i_0 \text{ and } I \cap (J \setminus j_0) = \emptyset; \\ 0 & \text{otherwise.} \end{cases}$$

for all  $I, J \subseteq [n]$ , where  $i_0 := \min(I)$ ,  $j_0 := \min(J)$ .

Recall here that  $E^0(\mathrm{H}_*(e_n; \mathbb{F}_p))$  is the associated graded of  $\mathrm{H}_*(e_n; \mathbb{F}_p)$  with respect to the Künneth filtration. In fact, we have a splitting of rings

$$\mathrm{H}_*(e_n; \mathbb{F}_p) \cong \pi_*(\mathrm{H}\mathbb{F}_p \wedge_{BP\langle n \rangle} e_n) \otimes_{\mathbb{F}_{p^n}} \mathrm{H}_*(BP\langle n \rangle; \mathbb{F}_{p^n}).$$

However, there are potential multiplicative extensions involving the left hand tensor factor when  $n > 2$ . These issues are addressed in Section 3.3. For example, the relation  $u^{p^i-1} f_{i,j} = 0$  always holds in homotopy. When  $n = 2$  we have the following result.

**Corollary 1.2.** *When  $n = 2$  we have that*

$$\pi_*(\mathrm{H}\mathbb{F}_p \wedge_{BP\langle 2 \rangle} e_2) \cong \frac{\mathbb{F}_{p^2}[[u_1]][u, \bar{p}, f_{1,2}]}{(u_1 u^{p-1}, u^{p^2-1}, u f_{1,2}, f_{1,2}^2)}.$$

The product structure that we end up with is an interesting and unfamiliar one to us. In characteristic  $p$ , divided power algebras frequently arise, and the structure we have here is similar. There is a particular element  $u$ , which is a unit in  $E_n$  but not in  $e_n$ . Many of the products that we have can be understood as divided products with respect to  $u$ . We can also identify the  $f_I$ 's as Toda brackets. We hope to understand this multiplicative structure better in future work.

**1.1. Outline.** In Section 2 we show that the KSS based on an  $E_3$ -algebra  $R$  in  $S$ -modules with coefficients on commutative  $S$ -algebras under  $R$  is multiplicative. We first recall the  $\Lambda_f$  construction of Mandell along with some of its properties. We also recall the relevant work of the second author from [Til16] for establishing that a spectral sequence is multiplicative. We then proceed to show that the spectral sequence is multiplicative by constructing a map of filtrations

$$\Lambda_\alpha(\mathrm{HF}_{p\bullet}, c_\bullet(e_n), \mathrm{HF}_{p\bullet}, c_\bullet(e_n)) \longrightarrow \Lambda_\mu(\mathrm{HF}_{p\bullet}, c_\bullet(e_n)).$$

This map of filtrations induces the relevant product on Tor and filters the product map of  $\mathrm{HF}_p \wedge_{BP} e_n$ . This is enough to show that in the KSS the differentials satisfy the Leibniz formula and apply Theorem 2.15 of Baker and Richter.

In Section 3 we compute the  $E_2$ -page of the KSS. We compute this Tor-group with it's product structure and various Massey products. After resolving a couple of extension problems, we are able to identify these Massey products with Toda brackets and derive the collapse of the spectral sequence. After we have this collapse result we can deduce that the target of this KSS splits as the tensor product

$$H_*(e_n; \mathbb{F}_p) \cong H_*(BP\langle n \rangle; \mathbb{F}_{p^n}) \otimes_{\mathbb{F}_{p^n}} \pi_*(\mathrm{HF}_p \wedge_{BP\langle n \rangle} e_n).$$

**1.2. Conventions.** We work with the model of spectra called  $S$ -modules and  $S$ -algebras established in [EKMM97]. In Section 2 we will use  $R$  to denote an  $E_n$ -algebra in  $S$ -modules whose underlying  $S$ -module is cofibrant. We will use  $A$  and  $B$  to denote commutative  $S$ -algebras that receive a map of  $E_1$ -algebras from  $R$ . Following Mandell, we will take  $R$ -modules to mean modules over a strictly associative  $S$ -algebra, denoted  $UR$ , which is homotopy equivalent to  $R$  in the category of  $S$ -modules. Similarly, all model categorical notions will take place in  $UR$ -modules instead of  $R$ -modules. This  $S$ -algebra  $UR$  is constructed in Section 2 of [Man12]. Also, after the introduction of  $\Lambda_\mu$  we will take  $-\wedge_R-$  to mean  $\Lambda_\mu$ . In particular, all of the constructions in Section 2.2 take place in this setting. We also make use of the phrase homotopy cofibrant  $R$ -module. Such a module  $M$  over  $R$  is, in light of the above convention, a module over  $UR$  which is homotopy equivalent to a cell or cofibrant  $UR$ -module in the sense of [EKMM97].

Our computation is concerned with various spectra that arise chromatically. For more information about the spectra  $BP$  and  $BP\langle n \rangle$  we recommend the modern classic [Rav86]. We also recommend [Rez98] for information on Morava  $E$ -theory. Specifically, we work with the Brown-Peterson spectrum  $BP$ , the truncated Brown-Peterson  $BP\langle n \rangle$ , connective Morava  $E$ -theory  $e_n$ . Their homotopy groups are

$$\begin{aligned} \pi_*(BP) &\cong \mathbb{Z}_{(p)}[v_1, v_2, \dots], \\ \pi_*(BP\langle n \rangle) &\cong \mathbb{Z}_{(p)}[v_1, v_2, \dots, v_n], \\ \pi_*(e_n) &\cong \mathbb{W}(\mathbb{F}_{p^n})[[u_1, u_2, \dots, u_{n-1}]] [u] \end{aligned}$$

where the degrees of elements are given by  $\deg(v_i) = 2(p^i - 1)$ ,  $\deg(u_i) = 0$ , and  $\deg(u) = 2$ . Here,  $\mathbb{W}(\mathbb{F}_{p^n})$  is the ring of  $p$ -typical Witt vectors over the field  $\mathbb{F}_{p^n}$ . This ring is isomorphic to the unique unramified degree  $n$  extension of the  $p$ -adic integers,  $\mathbb{Z}_p[\zeta_n]$  where  $\zeta_n$  is a primitive  $(p^n - 1)^{\mathrm{st}}$  root of unity.

Those familiar with these spectra will note that there are choices involved in this description. In particular, what exactly is meant by the class  $v_i$ ? Our choice of  $v_i$  is dictated by the requirement that  $\varphi(v_i) = u_i u^{p^i - 1}$  where

$$BP\langle n \rangle \xrightarrow{\varphi} E_n$$

is the map of associative  $S$ -algebras constructed in Lemma 2.11.

Lastly, recall that the sign convention for the Leibniz formula in chain complexes of graded modules is

$$\partial(ab) = \partial(a)b + (-1)^{|a|}a\partial(b)$$

where  $|a|$  is the total degree of the homogeneous element  $a$ . This is the sign convention for all Leibniz formulas for differentials in spectral sequences.

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## 2. THE MULTIPLICATIVITY OF THE KÜNNETH SPECTRAL SEQUENCE FOR $E_4$ -ALGEBRAS

In [Til16], it is shown that the Künneth spectral sequence

$$\mathrm{Tor}_s^{\pi_* R}(\pi_* A, \pi_* B)_t \Rightarrow \pi_{s+t}(A \wedge_R B)$$

is multiplicative when  $R$  is a commutative  $S$ -algebra and  $A$  and  $B$  are  $R$ -algebras. In particular, the symmetry

$$\tau : M \wedge_R N \longrightarrow N \wedge_R M$$

of the monoidal structure on  $R$ -modules is explicitly used. In our situation we do not have a symmetric monoidal structure on the category of  $BP$ -modules itself but on the homotopy category of  $BP$ -modules. However, this symmetric monoidal structure on the homotopy category is induced by an interchange map which does exist on the category of  $BP$ -modules. First we briefly recall the work of Mandell where he constructs a point set model for the relative smash products and various interchange operations. Then we discuss multiplicative filtrations and other results from [Til16]. We then show that our spectral sequence is multiplicative. Finally, we recall Theorem 2.15 from [BR08] which we will use to show that the KSS collapses.

**2.1. Monoidal structure on the category of  $BP$ -modules.** In [Man12], Mandell constructs point set level models for monoidal products and interchange operations

$$\Lambda_f : (R - \mathrm{mod})^m \longrightarrow R - \mathrm{mod}$$

given a map

$$f : X \longrightarrow \mathfrak{C}_{n-1}(m).$$

Here  $X$  is a space,  $\mathfrak{C}_{n-1}(m)$  is the  $m^{\mathrm{th}}$ -space of the  $n-1$  little cubes operad, and  $R$  is an  $E_n$ -algebra in the category of  $S$ -modules. He then uses this construction to show what extra structure the derived category of modules over an  $E_n$ -algebra has for  $n \in \{2, 3, 4\}$ . For example, after constructing the  $\Lambda_f$  in general, Mandell obtains a monoidal product by taking  $\mu$  to be the element  $([0, 1/2], [1/2, 1])$  in  $\mathfrak{C}_1(2)$  and defining

$$M \wedge_R N := \Lambda_\mu(M, N).$$

The homotopical properties of this construction are nuanced. Mandell works with  $UR$ -modules instead of  $R$ -modules. Where  $R$  is only assumed to be an  $E_1$ -algebra in  $S$ -modules,  $UR$  is a homotopy equivalent associative  $S$ -algebra and so has a nice point set model for its category of modules.  $UR$  has the universal property that the data of an operadic (in the sense of  $E_1$ -algebras)  $R$ -module is the same as the data of an actual  $UR$ -module, the details can be found in Section 2 of [Man12]. Therefore, following Mandell, by  $R$ -module we will implicitly mean  $UR$ -modules. Similarly, all notions of cofibrancy are in terms of the model structure on  $UR$ -modules.

The idea behind more general  $\Lambda_f$ 's is that  $f$  can be used to construct a  $UR$ - $U(R^m)$ -bimodule structure on  $UR \wedge X_+$  which can then be used to form

$$\Lambda_f(M_1, M_2, \dots, M_m) := UR \wedge X_+ \wedge_{U(R^m)} (M_1 \wedge_S M_2 \wedge_S \dots \wedge_S M_m).$$

These  $\Lambda_f$  can be used to construct a family of  $R$ -module structures which are homotopy invariant and natural in  $f$ . They are also homotopy invariant in the  $R$ -module coordinates when applied to homotopy cofibrant modules. Two of the main results of [Man12] are the following theorems which will be very useful to our construction of the multiplicative structure on the Künneth filtration.

**Theorem 2.1** (Theorem 1.5 of [Man12]). *For cofibrant  $R$ -modules  $M_1, \dots, M_j$  where  $j = j_1 + \dots + j_m$ , the natural map*

$$\Lambda_{f \circ (g_1, \dots, g_m)}(M_1, \dots, M_j) \longrightarrow \Lambda_f(\Lambda_{g_1}(M_1, \dots, M_{j_1}), \dots, \Lambda_{g_m}(M_{j-j_m+1}, \dots, M_j))$$

*is a weak equivalence.*

This theorem is necessary for understanding how to compose interchange operations. It is very useful for constructing the product structure on  $\Lambda_\mu(A, B)$  in the category of  $R$ -modules for two commutative algebras under  $R$ . In particular, it is useful for establishing various coherences like the associativity condition.

**Theorem 2.2** (Theorem 1.7 of [Man12]). *Let  $R$  be a  $\mathfrak{C}_n$ -algebra,*

$$f : X \longrightarrow \mathfrak{C}_{n-1}(m)$$

*a map and  $M_1, \dots, M_m$   $R$ -modules. If  $X$  is homotopy equivalent to a CW-complex and  $M_1, \dots, M_m$  are homotopy equivalent to cofibrant  $R$ -modules then  $\Lambda_f(M_1, \dots, M_m)$  is homotopy equivalent to a cofibrant  $R$ -module.*

We will be applying this theorem in the situation that the  $M_i$  are stages in the Künneth filtration and hence are cofibrant by construction. It is also evident that from the construction of  $\Lambda_f$  that the functor is enriched. Therefore, if we have a homotopy between  $f$  and  $g$  then we obtain a homotopy equivalence between  $\Lambda_f$  and  $\Lambda_g$  applied to the same collection of  $R$ -modules. Consider the case when  $\sigma$  is a path in  $\mathfrak{C}_{n-1}(2)$  between two different orders of multiplying, which exists for  $n > 2$ . In this case we obtain a zig-zag of homotopy equivalences

$$\Lambda_\mu(M, N) \longrightarrow \Lambda_\sigma(M, N) \longleftarrow \Lambda_{\mu\tau}(M, N) = \Lambda_\mu(N, M)$$

where  $\mu\tau$  is the element  $([1/2, 1], [0, 1/2])$  in  $\mathfrak{C}_{n-1}(2)$  when  $M$  and  $N$  are cofibrant  $R$ -modules. This homotopy equivalence descends to the homotopy category so that  $M \wedge_R N$  and  $N \wedge_R M$  are the same in the homotopy category of  $R$ -modules when  $R$  is an  $E_n$ -algebra for  $n \geq 3$ . We will take advantage of the existence of these maps

$$\tau : \Lambda_\mu(M, N) \longrightarrow \Lambda_\mu(N, M)$$

which are by construction independent of the cofibrant  $R$ -modules  $M$  and  $N$ .

We will use this map in Section 2.3 in order to construct our multiplicative filtration of interest. The coherences established in [Man12] all still hold in the filtered setting.

The following proposition will also be useful when showing that  $\Lambda_f$  applied to a filtration is also a filtration.

**Lemma 2.3.** *Suppose that each  $M_i$  is a cell  $UR$ -module and that*

$$\varphi : M_1 \hookrightarrow M'_1$$

*is a relative cell  $UR$ -module. Then the induced map*

$$\Lambda_f(\varphi) : \Lambda_f(M_1, M_2, \dots, M_n) \longrightarrow \Lambda_f(M'_1, M_2, \dots, M_n)$$

*is also a relative cell  $UR$ -module and hence a cofibration.*

This is obvious as smashing a cell  $UR$ -module with cell  $R'$ -modules produces cell  $UR \wedge R'$ -modules by Proposition 3.10 of [EKMM97]. This result implies that when the construction  $\Lambda_f$  is applied to a filtration that we also obtain a filtration. One uses the above along with the pushout product axiom in order to obtain the result.

**2.2. Multiplicative filtrations.** The material in this section is a brief recollection of necessary material from both [Til16] and [Til17]. We will first recall some more classical notions and then explain how to adapt them to our situation where we do not have a genuine symmetric monoidal structure.

**Definition 2.4.** A filtered spectrum or filtration is a sequence of cofibrations

$$\cdots \hookrightarrow Y_{i-1} \hookrightarrow Y_i \hookrightarrow Y_{i+1} \hookrightarrow \cdots.$$

We denote the single filtered object as  $Y_\bullet$ . The associated graded complex of a filtered spectrum  $Y_\bullet$  is the complex of spectra

$$\cdots \leftarrow Y_{i-1}/Y_{i-2} \leftarrow Y_i/Y_{i-1} \leftarrow Y_{i+1}/Y_i \leftarrow \cdots$$

denoted by  $E^0(Y_\bullet)$ . Our notation  $A \twoheadrightarrow B$  is an abbreviation for  $A \rightarrow \Sigma B$ .

Here we work with increasing filtrations exclusively. The Künneth filtration, which gives rise to the KSS, is such an increasing filtration. In fact, the construction in [Til16] of the Künneth filtration shows that it is a cellular filtration in the sense of [EKMM97]. The category of filtered  $R$ -modules has both a notion of smash products and of homotopies of maps.

**Definition 2.5.** The smash product of two filtrations  $X_\bullet$  and  $Y_\bullet$  is denoted by  $\Gamma_\bullet(X_\bullet, Y_\bullet)$ . The  $n$ th term in the filtration is

$$\Gamma_n(X_\bullet, Y_\bullet) := \operatorname{colim}_{i+j \leq n} X_i \wedge_R Y_j.$$

We will also denote iterated smash products of  $r$ -filtrations  $X_\bullet^1, X_\bullet^2, \dots, X_\bullet^r$  by  $\Gamma^r(X_\bullet^1, X_\bullet^2, \dots, X_\bullet^r)_\bullet$ . Here, the  $n$ th term in the filtration is

$$\Gamma_n^r(X_\bullet^1, X_\bullet^2, \dots, X_\bullet^r) := \bigcup_{\sum_{i=1}^r \alpha_i = n} X_{\alpha_1}^1 \wedge_R X_{\alpha_2}^2 \wedge_R \cdots \wedge_R X_{\alpha_r}^r.$$

If all of the  $X_\bullet^i$  are the same filtration, we will use the symbol  $\Gamma_\bullet^r$  or simply  $\Gamma_\bullet$  when  $r = 2$ .

Here we use  $\cup$  to denote a colimit. This notation is inspired by the case where we have sequential inclusion of cell complexes. The above definition takes the form

$$\Gamma_n(X_\bullet, Y_\bullet) = X_0 \wedge_R Y_n \cup_{X_0 \wedge_R Y_{n-1}} X_1 \wedge_R Y_{n-1} \cup \cdots \cup_{X_{n-1} \wedge_R Y_0} X_n \wedge_R Y_0$$

when  $X_\bullet$  and  $Y_\bullet$  are filtrations concentrated in positive degrees so that  $X_i = Y_i = *$   $\forall i < 0$ . This definition has the feature that

$$E^0(\Gamma_\bullet(X_\bullet, Y_\bullet))_n \simeq (E^0(X_\bullet) \otimes E^0(Y_\bullet))_n := \bigvee_{i+j=n} X_i/X_{i-1} \wedge_R Y_j/Y_{j-1}$$

where the tensor is the graded tensor product of complexes of spectra. The above definition makes sense also in the category of  $UR$ -modules where we use  $\Lambda_\mu$  instead of  $-\wedge_R-$  as defined in the previous subsection. For convenience, we will use  $\Lambda_\mu$  to denote the construction  $\Gamma_\bullet$  where every instance of  $-\wedge_R-$  is replaced by  $\Lambda_\mu$ . Similarly, we will take  $\Lambda_f$  to mean the obvious thing when applied to filtrations. We can show that  $\Lambda_f$  applied to a filtration is again a filtration by applying Lemma 2.3 along with the repeated application of the pushout product axiom. The need for using  $\Lambda_\mu$  as opposed to  $-\wedge_R-$  is forced by the fact that our category does not have an honest symmetric monoidal structure.

We need one more definition before we give the definition of multiplicative filtrations and our lifting theorem.

**Definition 2.6.** Let  $f_\bullet, g_\bullet : A_\bullet \rightarrow B_\bullet$  be two maps of filtered spectra. We call  $H_\bullet : \Gamma_\bullet(R \wedge I_\bullet, A_\bullet) \rightarrow B_\bullet$  a filtered homotopy from  $f_\bullet$  to  $g_\bullet$  if the following diagram commutes.

$$\begin{array}{ccccc}
 \Gamma_\bullet(R \wedge c(0_+)_\bullet, A_\bullet) \simeq A_\bullet & \xrightarrow{\quad f_\bullet \quad} & & & B_\bullet \\
 & \searrow & \Gamma_\bullet(R \wedge I_\bullet, A_\bullet) & \xrightarrow{\quad H_\bullet \quad} & \\
 \Gamma_\bullet(R \wedge c(1_+)_\bullet, A_\bullet) \simeq A_\bullet & \xrightarrow{\quad g_\bullet \quad} & & & 
 \end{array}$$

Here,  $I_\bullet$  is the filtered spectrum coming from the standard cellular structure on the unit interval and  $c(X)_\bullet$  is the constant filtration where every map is the identity.

In filtration 0 we have  $I_0 := \{0, 1\}_+ \simeq S^0 \vee S^0$ . In filtration  $n$  we have that  $I_n := I_+$  and the maps in the filtration are the obvious inclusions. A filtered homotopy induces a chain homotopy on the associated graded complex.

We can now give a definition of multiplicative filtration. Such filtrations will give rise to multiplicative spectral sequences in the same way that pairings of filtrations give rise to pairings of spectral sequences. Note that the actual structure is given by maps of filtrations. However, we only require that coherences hold in the homotopy category. As we will be applying homotopy invariant functors in order to obtain our spectral sequences, this is sufficient. Further, our methods of constructing multiplicative filtrations from [Til16] are only capable of constructing maps that are coherent up to homotopy.

**Definition 2.7.** We say that a filtration  $X_\bullet$  is multiplicative if there is a map of filtrations

$$\Gamma_\bullet(X_\bullet, X_\bullet) = \Gamma_\bullet \xrightarrow{\mu_\bullet} X_\bullet.$$

In particular, we require maps  $\mu_n : \Gamma_n \rightarrow X_n$  such that

$$\begin{array}{ccc}
 \Gamma_{n-1} & \xrightarrow{\quad} & \Gamma_n \\
 \mu_{n-1} \downarrow & & \downarrow \mu_n \\
 X_{n-1} & \xrightarrow{\quad} & X_n
 \end{array}$$

commutes. We also require that  $\mu_\bullet$  satisfy the obvious associativity condition up to filtered homotopy.

We will be interested in showing that the KSS applied to commutative  $S$ -algebras under  $BP$  is multiplicative. In particular, this implies that all differentials satisfy the Leibniz formula. Filtrations are frequently multiplicative when the object being filtered has a (potentially weak in the sense of  $A_\infty$ ) associative product structure.

To see that this implies the differentials satisfy the Leibniz formula consider a pairing of filtrations

$$\Gamma_\bullet(X_\bullet^1, X_\bullet^2) \rightarrow Y_\bullet.$$

Such a paring gives a map of spectral sequences. The general argument is as follows. Classes  $x_1 \in E_r(X_\bullet^1)$  and  $x_2 \in E_r(X_\bullet^2)$  with differentials  $d_r(x_i)$  respectively, are represented by maps of filtered spectra

$$\begin{array}{ccccccccccc} U(r, s_i, n_i) & & * & \longrightarrow & S^{n_i-1} & \longrightarrow & S^{n_i-1} & \longrightarrow & S^{n_i-1} & \longrightarrow & CS^{n_i-1} & \longrightarrow & CS^{n_i-1} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X_\bullet^i & & X_{s_i-r-1}^i & \longrightarrow & X_{s_i-r}^i & \longrightarrow & \cdots & \longrightarrow & X_{s_i-1}^i & \longrightarrow & X_{s_i}^i & \longrightarrow & X_{s_i+1}^i \end{array}$$

We then smash these two maps of filtered spectra

$$\tilde{x}_1 \wedge \tilde{x}_2 : \Gamma_\bullet(U(r, s_1, n_1)_\bullet, U(r, s_2, n_2)_\bullet)_\bullet \longrightarrow \Gamma_\bullet(X_\bullet^1, X_\bullet^2)_\bullet.$$

The isomorphism of associated graded complexes

$$\pi_*(E^0(\Gamma_\bullet(U(r, s_1, n_1)_\bullet, U(r, s_2, n_2)_\bullet)_\bullet)) \cong \pi_*(E^0(U(r, s_1, n_1)_\bullet)) \bigotimes \pi_*(E^0(U(r, s_2, n_2)_\bullet))$$

provides a simple description of the spectral sequence associated with  $\Gamma_\bullet(U(r, s_1, n_1)_\bullet, U(r, s_2, n_2)_\bullet)_\bullet$ . To obtain the desired formula we look at the differentials in this spectral sequence and push them forward along the composition

$$\mu_\bullet \circ (\tilde{x}_1 \wedge \tilde{x}_2) : \Gamma_\bullet(U(r, s_1, n_1)_\bullet, U(r, s_2, n_2)_\bullet)_\bullet \longrightarrow \Gamma_\bullet(X_\bullet^1, X_\bullet^2)_\bullet \longrightarrow Y_\bullet.$$

The Künneth filtration can be thought of as a cellular filtration. It is gotten by examining a projective resolution of the homotopy groups. The explicit details can be found in Chapter 4.5 [EKMM97] and Sections 2 and 4 of [Til16]. The particulars that are necessary are that the filtration is free and exact in the sense of [Til16]. This is always the case when the filtration is constructed from a free and exact resolution. A filtration is free when its associated graded complex is a free complex and a filtration being exact implies that its associated graded complex is exact. For more details see Section 2.1 of [Til16]. This exactness and freeness is necessary in order to apply the following result from.

**Theorem 2.8** (Tilson, Thm 1 [Til16]). *Suppose that we have a map  $f : Y \rightarrow A$  of  $R$ -modules, an exact filtration  $A_\bullet \subset A$ , and a free and exhaustive filtration  $Y_\bullet \subset Y$ . Also, suppose that there exist  $f_{-1} : Y_{-1} \rightarrow A_{-1}$  such that*

$$\begin{array}{ccc} Y_{-1} & \xrightarrow{\quad} & Y \\ f_{-1} \downarrow & & \downarrow f \\ A_{-1} & \xrightarrow{\quad} & A \end{array}$$

*commutes. Then there is a map of filtrations  $Y_i \xrightarrow{f_i} A_i$  such that  $\text{colim } f_i \simeq f$  under the equivalences  $\text{colim } Y_i \simeq Y$  and  $\text{colim } A_i \simeq A$ . Furthermore, the lift  $f_\bullet$  of  $f$  is unique up to homotopy of filtered modules, in the sense of Definition 2.6.*

Now we wish to show that the filtration  $\Gamma_\bullet(A_\bullet, c(B)_\bullet)$  is multiplicative where  $A_\bullet$  is the Künneth filtration of  $A$  and both  $A$  and  $B$  are  $R$ -algebras, and  $R$  is assumed to be a commutative  $S$ -algebra for the time being. As  $A_\bullet$  is a free and exact filtration we can apply the above theorem to the  $R$ -algebra structure map of  $A$

$$\mu^A : A \wedge_R A \longrightarrow A.$$



Here,  $Y_\bullet = \Gamma_\bullet(A_\bullet, A_\bullet)$  is easily seen to be free. We then have that  $A_\bullet$  is a multiplicative filtration with

$$\mu_\bullet^A : \Gamma_\bullet(A_\bullet, A_\bullet) \longrightarrow A_\bullet$$

which we can use in conjunction with the  $R$ -algebra structure of  $B$  to obtain the product map

$$\Gamma_\bullet(A_\bullet, c(B)_\bullet, A_\bullet, c(B)_\bullet) \xrightarrow{1 \wedge \tau \wedge 1} \Gamma_\bullet(A_\bullet, A_\bullet, c(B)_\bullet, c(B)_\bullet) \xrightarrow{\mu_\bullet^A \wedge \mu_\bullet^B} \Gamma_\bullet(A_\bullet, c(B)_\bullet).$$

Clearly, the symmetry  $\tau$  on the category of  $R$ -modules is essential. In our situation,  $R$  is not itself a commutative  $S$ -algebra. It is shown that  $BP$  is an  $E_4$ -algebra in [BM13]. Mandell showed in [Man12] that this is enough to have a symmetric monoidal structure on the derived category of modules. The above definitions can be made in the category of  $BP$ -modules where we take every occurrence of the symbol  $-\wedge_R-$  to mean  $\Lambda_\mu$  from Section 2.1. The coherences established by Mandell for this “monoidal” structure still hold in the filtered setting. Instead of having the classical associativity diagram which commutes up to homotopy, we have a more complicated diagram which involves backwards weak equivalences. These backwards weak equivalences are all natural though. See Proposition 2.10 for a more precise statement. Also note that Theorem 2.8 itself works in this setting of  $UR$ -modules with  $\Lambda_\mu$  since exactness and freeness are homotopical notions.

**2.3. Modules and Algebras under  $BP$ .** In order to show that the spectral sequence is multiplicative we have to have a  $BP$ -algebra structure map to lift. In this subsection we construct that map as well as the necessary twist map. After doing this, we construct the lifts to the filtered setting.

In ordinary ring theory, when working with a map of associative algebras

$$R \longrightarrow A$$

it is not the case that  $A$  inherits an  $R$ -algebra structure as there may be elements of  $R$  that are not central in  $A$ . However, if  $A$  is commutative then every element of  $A$  is central. A similar argument works here so that both  $H\mathbb{F}_p$  and  $e_n$  are algebras under  $BP$ .

**Lemma 2.9.** *Every map of  $S$ -algebras*

$$f : R \longrightarrow A$$

*with codomain a commutative  $S$ -algebra induces an  $R$ -algebra structure on  $A$ .*

We can form a version of the above result using the  $\Lambda_\mu$  construction instead. While we see that  $-\wedge_{BP}-$  when applied to commutative  $S$ -algebras under  $BP$  has a  $BP$ -module structure, the same can not be said of  $BP$ -modules, of which the Künneth filtration is comprised.

**Proposition 2.10.** *Given a commutative  $S$ -algebra  $A$  and an  $E_n$ -algebra  $R$  with a map of  $E_1$ -algebras in  $S$ -modules*

$$R \longrightarrow A$$

*we have the product map*

$$\Lambda_\mu(A, A) \longrightarrow A$$

*in the category of  $R$ -modules. This product is associative in the sense that*

$$\begin{array}{ccccccc} \Lambda_\mu(\Lambda_\mu(A, A), A) & \longleftarrow & \Lambda_{\mu \circ_1 \mu}(A, A, A) & \longrightarrow & \Lambda_\alpha(A, A, A) & \longleftarrow & \Lambda_{\mu \circ_2 \mu}(A, A, A) \longrightarrow \Lambda_\mu(A, \Lambda_\mu(A, A)) \\ \downarrow & & & & & & \downarrow \\ \Lambda_\mu(A, A) & \longrightarrow & A & \longleftarrow & & & \Lambda_\mu(A, A) \end{array}$$

*commutes.*

Here,  $\alpha$  is a path in  $\mathfrak{C}_{n-1}(3)$  between the two different ways of associating a product. The top row of the diagram is referred to as (1.6) in [Man12]. Diagram (1.6) follows Theorem 1.5 which explains how to compose such interchange operations and they are both used to establish coherences in the derived category of  $R$ -modules. Further, Mandell has established that each map in the top row induces a weak equivalence when applied to homotopy cofibrant  $R$ -modules.

*Proof.* The construction  $\Lambda_\mu$  is defined by the coequalizer diagram

$$UR\mu \wedge U(R^2) \wedge A \wedge A \rightrightarrows UR\mu \wedge A \wedge A$$

Recall that  $UR\mu$  is the  $UR-U(R^2)$ -bimodule  $UR \wedge *_{+}$  where the right  $U(R^2)$ -module structure comes from the inclusion of the point  $\mu \in \mathfrak{C}_{n-1}(2)$ . What we then wish to construct is a map of coequalizer diagrams

$$\begin{array}{ccc} UR\mu \wedge U(R^2) \wedge A \wedge A & \rightrightarrows & UR\mu \wedge A \wedge A \\ \downarrow & & \downarrow \\ UR\mu \wedge UR \wedge A & \rightrightarrows & UR\mu \wedge A. \end{array}$$

The right  $U(R^2)$ -module structure on  $UR\mu$  is defined using the multiplication map

$$U(R^2) \longrightarrow UR.$$

Thus we have a commutative diagram

$$\begin{array}{ccc} UR\mu \wedge U(R^2) & \longrightarrow & UR\mu \\ \downarrow & & \downarrow \\ UR\mu \wedge UR & \longrightarrow & UR\mu. \end{array}$$

The left  $U(R^2)$ -module structure of  $A \wedge A$  is the composite of

$$f : U(R^2) \wedge A \wedge A \longrightarrow UR \wedge UR \wedge A \wedge A \longrightarrow UR \wedge A \wedge UR \wedge A \longrightarrow A \wedge A \wedge A \wedge A$$

and the multiplication of  $A$  as a commutative  $S$ -algebra smashed with itself. The map  $f$  is a composition of

$$U(R \wedge R) \longrightarrow UR \wedge UR,$$

which is referred to as 1.2 in [Man12], the natural equivalences

$$UR \longrightarrow R,$$

the map from  $R$  to  $A$  that we began with, and the symmetry in the underlying category of  $S$ -modules. Therefore the diagram

$$\begin{array}{ccc} U(R^2) \wedge A \wedge A & \longrightarrow & A \wedge A \\ \downarrow & & \downarrow \\ UR \wedge A & \longrightarrow & A \end{array}$$

commutes since  $A$  is a commutative and associative  $S$ -algebra. We now have a map of coequalizer diagrams. This naturally induces a map

$$\Lambda_\mu(A, A) \longrightarrow A.$$

To see that the associativity condition mentioned is satisfied one can make a similar argument. As  $A$  is an associative  $S$ -algebra we can write down a product map

$$\Lambda_\alpha(A, A, A) = (UR \wedge I_+) \wedge_{U(R^3)} (A \wedge A \wedge A) \longrightarrow A$$

by examining coequalizer diagrams as we did above. We can also use the functoriality of the above construction to write down

$$\Lambda_\mu(\Lambda_\mu(A, A), A) \longrightarrow \Lambda_\mu(A, A)$$

and

$$\Lambda_\mu(A, \Lambda_\mu(A, A)) \longrightarrow \Lambda_\mu(A, A).$$

The desired diagram commutes due to the discussion in the beginning of Section 4.5 of [Man12].  $\square$

Note that the associativity condition is slightly less than what is required by our definition of multiplicative filtration. This won't cause any technical trouble as we are applying functors that take weak equivalences to isomorphisms. This means that any geometric zig-zag will be transformed into an honest composite in the setting where we do any algebraic computations. The weak equivalences are natural in the module coordinates and so they produces isomorphisms that are natural in these coordinates. Further, the main use of the multiplicative structure of a spectral sequence is to establish the Leibniz formula and for this we only use the pairing

$$\Lambda_\mu(A_\bullet, A_\bullet) \longrightarrow A_\bullet.$$

In order to apply the above results, we need maps of associative  $S$ -algebras or  $E_1$ -algebras

$$BP \longrightarrow A.$$

When  $A$  is  $\mathrm{HF}_p$  this is clear as we can take the composition of 0th Postnikov sections and the reduction mod  $p$  map

$$BP \longrightarrow P_0(BP) = \mathrm{HZ}_{(p)} \longrightarrow \mathrm{HF}_p$$

which is a map of associative algebras. Obtaining the map to connective Morava  $E$ -theory is more difficult and relies on the work of Lazarev in [Laz03], Angeltveit in [Ang08], Rognes in [Rog08], and many others. Once we have a map of  $A_\infty$ -ring spectra or associative  $S$ -algebras

$$BP \longrightarrow E_n$$

we can easily construct the map to  $e_n$ . The following result is not original and is proved by stitching together various results in the literature. We don't know of a proper reference and so we provide the result and the argument here.

**Lemma 2.11.** *There is a map of associative  $S$ -algebras*

$$BP \longrightarrow e_n$$

*which takes the class  $v_i$  to  $u_i u^{p^i-1}$  for  $i < n$ ,  $u^{p^n-1}$  for  $i = n$ , and 0 otherwise.*

*Proof.* By work of Lazarev in [Laz03] and Angeltveit in [Ang08], we have a map of associative  $S$ -algebras

$$BP \longrightarrow E(n)$$

to periodic Johnson-Wilson theory. Their methods also extend this along the  $I$ -adic completion. The completion step can also be viewed as composing the above map with

$$E(n) \longrightarrow L_{K(n)} E(n)$$

which is known to preserve monoidal structures.

We obtain the map of  $E_n$  using work of Rognes from [Rog08]. Rognes shows in Proposition 5.4.9 that the map

$$\widehat{E(n)} \longrightarrow E_n$$

is a faithful Galois extension. In his proof, he shows that

$$\widehat{E(n)} \simeq E_n^{hK}$$

for a particular subgroup  $K$  of the extended Morava stabilizer group. The computation of Rognes provides a map of commutative  $S$ -algebras that takes the class  $v_i$  to the desired element in  $\pi_* E_n$ . We now have a composite

$$BP \longrightarrow E(n) \longrightarrow \widehat{E(n)} \simeq E_n^{hK} \longrightarrow E_n$$

of maps of associative  $S$ -algebras which induces the desired map of rings on homotopy.

Now that we have the map

$$BP \longrightarrow E_n$$

we can lift this to the connective cover  $e_n$  via the following standard procedure. We can construct  $e_n$  as an associative  $S$ -algebra by attaching cells to  $E_n$  to kill all of the nonnegative homotopy groups in the category of associative  $S$ -algebras. This gives a map

$$E_n \longrightarrow \tau_{<0} E_n$$

of associative  $S$ -algebras which is an isomorphism in  $\pi_i$  for  $i < 0$ . The fiber of this is then connective and an associative  $S$ -algebra as fibers in  $S$ -algebras can be computed in the underlying category of  $S$ -modules. The spectrum  $BP$  is connective, therefore the composition

$$BP \longrightarrow E_n \longrightarrow \tau_{<0} E_n$$

is nullhomotopic. Thus we have our desired lift

$$BP \longrightarrow e_n$$

in the category of  $S$ -algebras. □

We are now in a situation to apply the above result Theorem 2.8 to the map of  $S$ -algebras

$$BP \longrightarrow \mathrm{HF}_p.$$

We let  $\mathrm{HF}_{p\bullet}$  denote the Künneth filtration of  $\mathrm{HF}_p$  in the category of  $BP$ -modules constructed as in Section 2 of [Til16].

**Proposition 2.12.** *The filtration  $\Lambda_\mu(\mathrm{HF}_{p\bullet}, c_\bullet(e_n))$  is multiplicative.*

We give the proof in this specific example, but the argument generalizes to any algebra over an  $E_3$ -algebra in  $S$ -modules.

*Proof.* To simplify the notation, for the duration of this proof we set  $\mathcal{F} := \mathrm{HF}_p$  and  $\mathcal{E} := e_n$ . Similarly,  $\mathcal{F}_\bullet$  will denote the free and exact filtration of  $\mathrm{HF}_p$  that comes from the Koszul complex and  $\mathcal{E}_\bullet$  will denote the constant filtration of  $e_n$ . We already have the maps  $\mathcal{F} \wedge_{BP} \mathcal{F} \longrightarrow \mathcal{F}$  and  $\mathcal{E} \wedge_{BP} \mathcal{E} \longrightarrow \mathcal{E}$ . These maps extend to both  $\Lambda_\mu(\mathcal{F}, \mathcal{F}) \longrightarrow \mathcal{F}$  and  $\Lambda_\mu(\mathcal{E}, \mathcal{E}) \longrightarrow \mathcal{E}$ .

We now wish to lift these to the filtrations  $\mathcal{F}_\bullet := \mathrm{HF}_{p\bullet}$  and  $\mathcal{E}_\bullet := c_\bullet(e_n)$  of  $\mathcal{F}$  and  $\mathcal{E}$  in  $BP$ -modules, respectively. We now lift the map using Theorem 2.8 applied to the filtration  $\Lambda_\mu(\mathcal{F}_\bullet, \mathcal{F}_\bullet)$  and the map

$$\Lambda_\mu(\mathcal{F}, \mathcal{F}) \longrightarrow \mathcal{F}.$$

To apply the result, note that since  $\mathcal{F}_\bullet$  is a free filtration, meaning that the associated graded complex of  $BP$ -module spectra is levelwise free, we also have that  $\Gamma_\bullet(\mathcal{F}_\bullet, \mathcal{F}_\bullet)$  is levelwise free. The argument

is the same as that when applied to  $-\wedge_R-$  as opposed to  $\Lambda_\mu$ , one just recalls that freeness here is in the category of  $UR$ -modules. This still models the derived smash product of  $BP$ -modules as each spectrum in the filtration  $\mathcal{F}_\bullet$  is cofibrant by construction.

Now we have our desired map

$$\Lambda_\mu(\mathcal{F}_\bullet, \mathcal{F}_\bullet) \longrightarrow \mathcal{F}_\bullet.$$

We can construct the multiplicative structure on the filtration  $\Lambda_\mu(\mathcal{F}_\bullet, \mathcal{E}_\bullet)$  as follows. We first have the morphism

$$\Lambda_{\mu \circ (\mu \wedge \mu)}(\mathcal{F}_\bullet, \mathcal{E}_\bullet, \mathcal{F}_\bullet, \mathcal{E}_\bullet) \longrightarrow \Lambda_\mu(\Lambda_\mu(\mathcal{F}_\bullet, \mathcal{E}_\bullet), \Lambda_\mu(\mathcal{F}_\bullet, \mathcal{E}_\bullet))$$

which is a weak equivalence by Theorem 1.5 of [Man12]. We now choose a path

$$\alpha : I \longrightarrow \mathfrak{C}_3(4)$$

from  $\mu \circ (\mu \wedge \mu)$  and  $\mu \circ (\mu \wedge \mu) \circ (1 \wedge \tau \wedge 1)$ . This then gives a zig-zag of weak equivalences

$$\Lambda_{\mu \circ (\mu \wedge \mu)}(\mathcal{F}_\bullet, \mathcal{E}_\bullet, \mathcal{F}_\bullet, \mathcal{E}_\bullet) \longrightarrow \Lambda_\alpha(\mathcal{F}_\bullet, \mathcal{E}_\bullet, \mathcal{F}_\bullet, \mathcal{E}_\bullet) \longleftarrow \Lambda_{\mu \circ (\mu \wedge \mu)}(\mathcal{F}_\bullet, \mathcal{F}_\bullet, \mathcal{E}_\bullet, \mathcal{E}_\bullet)$$

since we have the following isomorphism of  $S$ -modules

$$\Lambda_{\mu \circ (\mu \wedge \mu) \circ (1 \wedge \tau \wedge 1)}(\mathcal{F}_\bullet, \mathcal{E}_\bullet, \mathcal{F}_\bullet, \mathcal{E}_\bullet) \cong \Lambda_{\mu \circ (\mu \wedge \mu)}(\mathcal{F}_\bullet, \mathcal{F}_\bullet, \mathcal{E}_\bullet, \mathcal{E}_\bullet).$$

This uses Theorem 1.5 of [Man12] again, see Theorem 2.1. We now can use the structure maps

$$\Lambda_\mu(\mathcal{F}_\bullet, \mathcal{F}_\bullet) \longrightarrow \mathcal{F}_\bullet$$

as well as

$$\Lambda_\mu(\mathcal{E}_\bullet, \mathcal{E}_\bullet) \longrightarrow \mathcal{E}_\bullet.$$

Thus we have the map

$$\Lambda_{\mu \circ (\mu \wedge \mu)}(\mathcal{F}_\bullet, \mathcal{F}_\bullet, \mathcal{E}_\bullet, \mathcal{E}_\bullet) \longrightarrow \Lambda_\mu(\mathcal{F}_\bullet, \mathcal{E}_\bullet).$$

We then apply  $\pi_*$  to the composite of

$$\Lambda_\mu(\Lambda_\mu(\mathcal{F}_\bullet, \mathcal{E}_\bullet), \Lambda_\mu(\mathcal{F}_\bullet, \mathcal{E}_\bullet)) \longleftarrow \Lambda_{\mu \circ (\mu \wedge \mu)}(\mathcal{F}_\bullet, \mathcal{E}_\bullet, \mathcal{F}_\bullet, \mathcal{E}_\bullet) \longrightarrow \Lambda_\alpha(\mathcal{F}_\bullet, \mathcal{E}_\bullet, \mathcal{F}_\bullet, \mathcal{E}_\bullet)$$

with

$$\Lambda_\alpha(\mathcal{F}_\bullet, \mathcal{E}_\bullet, \mathcal{F}_\bullet, \mathcal{E}_\bullet) \longleftarrow \Lambda_{\mu \circ (\mu \wedge \mu)}(\mathcal{F}_\bullet, \mathcal{F}_\bullet, \mathcal{E}_\bullet, \mathcal{E}_\bullet) \longrightarrow \Lambda_\mu(\Lambda_\mu(\mathcal{F}_\bullet, \mathcal{F}_\bullet), \Lambda_\mu(\mathcal{E}_\bullet, \mathcal{E}_\bullet))$$

and the induced product map

$$\Lambda_\mu(\Lambda_\mu(\mathcal{F}_\bullet, \mathcal{F}_\bullet), \Lambda_\mu(\mathcal{E}_\bullet, \mathcal{E}_\bullet)) \longrightarrow \Lambda_\mu(\mathcal{F}_\bullet, \mathcal{E}_\bullet).$$

Note that each of the maps involving four filtrations is a natural weak equivalence for each stage in the filtration. After applying homotopy this gives the map

$$\pi_*(\Lambda_\mu(\Lambda_\mu(\mathcal{F}_\bullet, \mathcal{E}_\bullet), \Lambda_\mu(\mathcal{F}_\bullet, \mathcal{E}_\bullet))) \longrightarrow \pi_*(\Lambda_\mu(\mathcal{F}_\bullet, \mathcal{E}_\bullet))$$

which induces the desired pairing on the spectral sequence.  $\square$

**2.4. Massey Products in the Künneth spectral sequence.** Here we review some material from [BR08]. The results of this section are due to Baker, Richter and Kochman. Our only contribution is the observation that they apply in our setting, which is obvious once it is known that the relevant KSS is multiplicative. Baker and Richter adapted work of Kochman from [Koc96] to the setting of a multiplicative KSS. Their work shows how to relate Massey products in Tor to Toda brackets in the target of the spectral sequence. We recall briefly some of the relevant definitions and then we will state their Theorem B.2 from [BR08].

There is the standard convention that  $\hat{a} = (-1)^{|a|+1}$  where  $|a|$  is the total degree of the homology class  $a$  in case we are working with graded chain complexes, see Appendix 1 of [Rav86] where the

notation  $\bar{a}$  is used instead. We also use the notation  $[a]$  to denote the homology or homotopy class of a cycle  $a \in A$  or a map

$$a : S^n \longrightarrow E$$

depending on the context.

**Definition 2.13.** Let  $[x], [y], [z] \in H_*(A)$  where  $A$  is a DGA and  $[x][y] = 0$ ,  $[y][z] = 0$  in  $H_*(A)$ . Then the Massey product  $\langle [x], [y], [z] \rangle$  is defined to be the set  $\{\hat{s}z + \hat{x}t \mid \partial(s) = \hat{x}y \text{ and } \partial(t) = \hat{y}z\}$ .

We call the data  $\{s, t, x, y, z\}$  along with their boundaries a defining system. The indeterminacy of this Massey product is given by

$$[x]H_*(A) \oplus H_*(A)[z]$$

for suitable degrees.

For homotopy groups of ring spectra there is the similar notion of Toda brackets. As there is a product structure to take advantage of, the definitions are remarkably similar. Therefore it is not surprising that they are related.

**Definition 2.14.** Let  $[a], [b], [c] \in \pi_*E$  where  $E$  is an  $R$ -ring spectrum and  $[a][b] = 0$ ,  $[b][c] = 0$  in  $\pi_*E$ . Then the Toda bracket  $\langle [a], [b], [c] \rangle$  is defined to be the set  $\{g_{ab}c + ag_{bc}\}$  where

$$g_{ij} : D^{|i|+|j|+1} \longrightarrow E$$

is a nullhomotopy of the product

$$i \wedge j : S^{|i|+|j|} \cong S^{|i|} \wedge S^{|j|} \longrightarrow E \wedge E \xrightarrow{\mu} E.$$

Note that we only use the existence of a product on  $E$ . This setting can be adapted to work in modules over any ring spectrum, such as  $BP$ , which is what we do here.

We now have the result of Baker and Richter which allows us to compute differentials by relating Massey products and Toda brackets.

**Theorem 2.15** (Theorem B.2 of [BR08]). *Assume that the following conditions hold in the KSS*

$$\mathrm{Tor}^{R*}(A_*, B_*) \Rightarrow \pi_*(A \wedge_R B).$$

- The elements  $[x], [y], [z] \in E_r$  are permanent cycles which converge to elements  $\xi_1, \xi_2, \xi_3$  in  $\pi_*(A \wedge_R B)$  respectively.
- The Massey product  $\langle [x], [y], [z] \rangle$  is defined in  $E_{r+1}$ .
- The Toda bracket  $\langle \xi_1, \xi_2, \xi_3 \rangle$  is defined in  $\pi_*(A \wedge_R B)$ .
- If  $\{s, t, x, y, z\}$  is a defining system for  $\langle [x], [y], [z] \rangle$  then there are no nonzero crossing differentials for the differentials  $d_r(s) = \hat{x}y$  and  $d_r t = \hat{y}z$ .

Then  $\langle [x], [y], [z] \rangle$  is a set of permanent cycles which converge to elements of  $\langle \xi_1, \xi_2, \xi_3 \rangle$ .

The only part of multiplicativity used in the proof is the pairing of the filtration with itself. The notion of crossing differential is slightly technical. Let  $y \in E_r^{a, n-a}$  with  $d_r y \neq 0$ . We say that  $y$  has a *crossing differential* if there is an element  $y' \in E_{r'}^{a', n-a'}$  with  $d_{r'} y' \neq 0$  such that  $a < a'$  and  $a + r > a' + r'$ . If one draws a spectral sequence one sees that the differentials cross each other. In our situation, there will be no nonzero crossing differentials for degree reasons.

### 3. THE $E_2$ -PAGE OF THE KÜNNETH SPECTRAL SEQUENCE AND DIFFERENTIALS

In this section we compute the  $E_2$ -page of the Künneth spectral sequence

$$\mathrm{Tor}_s^{BP_*}(\mathrm{H}_*(BP; \mathbb{F}_p), e_{n*})_t \Rightarrow \mathrm{H}_{s+t}(e_n; \mathbb{F}_p)$$

as an algebra. We then use this result to show that in small height the spectral sequence collapses. This is enough to compute the homology of connective Morava  $E$ -theory in small height, up to multiplicative extensions. We will resolve the multiplicative extensions in the Section 3.3.

**3.1. The  $E_2$ -page as an algebra.** Our first step in our computation is the following lemma.

**Lemma 3.1.** *There is an isomorphism of bigraded algebras*

$$\mathrm{Tor}_*^{BP_*}(\mathrm{H}_*(BP; \mathbb{F}_p), e_{n*})_* \cong \mathrm{H}_*(BP; \mathbb{F}_p) \otimes_{\mathbb{F}_p} \mathrm{Tor}_*^{BP\langle n \rangle_*}(\mathbb{F}_p, e_{n*}) \otimes_{\mathbb{F}_p} E_{\mathbb{F}_p}[\bar{v}_{n+1}, \bar{v}_{n+2}, \dots].$$

where  $E_{\mathbb{F}_p}[\bar{v}_{n+1}, \bar{v}_{n+2}, \dots]$  is the exterior algebra over  $\mathbb{F}_p$  on the indicated generators.

*Proof.* First, note that the Hurewicz map  $BP_* \rightarrow \mathrm{H}_*(BP; \mathbb{F}_p)$  is zero except in degree 0 where it is surjective. This implies that the  $BP_*$  module structure on  $\mathrm{H}_*(BP; \mathbb{F}_p)$  is trivial, in the sense that  $\mathrm{H}_*(BP; \mathbb{F}_p) \cong \oplus \mathbb{F}_p$  as a  $BP_*$ -module. Thus we have that

$$\mathrm{Tor}_*^{BP_*}(\mathrm{H}_*(BP; \mathbb{F}_p), e_{n*})_* \cong \mathrm{H}_*(BP; \mathbb{F}_p) \otimes \mathrm{Tor}_*^{BP_*}(\mathbb{F}_p, e_{n*})_*.$$

Further, note that the map  $BP \rightarrow e_n$  factors through  $BP\langle n \rangle$ . In other words, the elements  $\bar{v}_{n+k}$  are sent to zero for  $k \geq 0$ . Thus, computing  $\mathrm{Tor}_*^{BP_*}(\mathbb{F}_p, e_{n*})_*$  by resolving  $\mathbb{F}_p$  over  $BP_*$  immediately yields an isomorphism

$$\mathrm{Tor}_*^{BP_*}(\mathbb{F}_p, e_{n*})_* \cong \mathrm{Tor}_*^{BP\langle n \rangle_*}(\mathbb{F}_p, e_n) \otimes_{\mathbb{F}_p} E_{\mathbb{F}_p}[\bar{v}_{n+1}, \bar{v}_{n+2}, \dots].$$

□

The homology of  $BP$  and  $BP\langle n \rangle$  are

$$\mathrm{H}_*(BP; \mathbb{F}_p) \cong \begin{cases} \mathbb{F}_2[\xi_1^2, \xi_2^2, \dots] & \text{if } p = 2; \\ \mathbb{F}_p[\xi_1, \xi_2, \dots] & \text{if } p \neq 2 \end{cases}$$

and

$$\mathrm{H}_*(BP\langle n \rangle; \mathbb{F}_p) \cong \begin{cases} \mathbb{F}_2[\xi_1^2, \xi_2^2, \dots, \xi_{n+1}^2, \xi_{n+2}, \xi_{n+3}, \dots] & \text{if } p = 2; \\ \mathbb{F}_p[\xi_1, \xi_2, \dots] \otimes E(\tau_{n+1}, \tau_{n+2}, \dots) & \text{if } p \neq 2. \end{cases}$$

When we take coefficients in  $\mathbb{F}_{p^n}$  instead of  $\mathbb{F}_p$  this only has the effect adjoining roots of unity. It remains to compute  $\mathrm{Tor}_*^{BP\langle n \rangle_*}(\mathbb{F}_p, e_{n*})$ . We are going to use the notation  $[n] := \{1, \dots, n\}$  for  $n \in \mathbb{N}$ , and we further define  $w : [n] \rightarrow \mathbb{N}$  as  $w(i) := p^i - 1$ . Moreover, recall that  $e_{n*} = \mathbb{W}(\mathbb{F}_{p^n})[[u_1, u_2, \dots, u_{n-1}]] [u]$ .

Let  $A$  be the exterior  $e_{n*}$ -algebra with generators  $f'_I$  for each  $I \subset [n]$  with  $\#I \geq 2$ . We endow  $A$  with a bigrading by setting  $\deg f'_I := (\#I - 1, 2(\sum_{i \in I \setminus \min(I)} w(i)))$ . Here, the first component of the grading is to be interpreted as a homological grading, and the second component is an internal grading. Further, let  $\mathfrak{a} \subset A$  be the ideal generated by the following polynomials:

$$\begin{aligned} & u^{w(i)} u_i \quad \text{for } 0 \leq i \leq n; \\ & u^{w(\min I)} f'_I \quad \text{for } I \subseteq [n]; \\ & u_a f'_{I \cup b} - u_b f'_{I \cup a} \quad \text{for } I \subseteq [n], \ a, b \in [n] \text{ with } a, b < \min(I) \end{aligned}$$

and

$$f'_I \cdot f'_J - \begin{cases} (-1)^{m(I \setminus i_0, J)} u_{i_0} f'_{(I \setminus i_0) \cup J} & \text{if } i_0 \geq j_0 \text{ and } (I \setminus i_0) \cap J = \emptyset; \\ (-1)^{m(I, J \setminus j_0)} u_{j_0} f'_{I \cup (J \setminus j_0)} & \text{if } j_0 \geq i_0 \text{ and } I \cap (J \setminus j_0) = \emptyset; \\ 0 & \text{otherwise.} \end{cases}$$

for all  $I, J \subset [n]$ , where  $i_0 := \min(I)$ ,  $j_0 := \min(J)$ , and  $m(A, B) := \#\{(a, b) \in A \times B \mid a > b\}$ .

**Theorem 3.2.**  $\text{Tor}_*^{BP\langle n \rangle_*}(\mathbb{F}_p, e_{n*})$  is isomorphic to  $E_{\mathbb{F}_p}(\bar{p}) \otimes_{\mathbb{F}_p} A/\mathfrak{a}$  as an  $e_{n*}$ -algebra.

*Proof.* Recall that  $BP\langle n \rangle_* = \mathbb{Z}_{(p)}[v_1, \dots, v_n]$ . The Koszul complex  $\mathcal{K} := K(p, v_1, \dots, v_n)$  over  $BP\langle n \rangle_*$  resolves its residue field  $\mathbb{F}_p$ . Hence it holds that

$$\text{Tor}_*^{BP\langle n \rangle_*}(\mathbb{F}_p, e_{n*}) \cong H_*(\mathcal{K} \otimes_{BP\langle n \rangle_*} e_{n*}).$$

Further,  $BP\langle n \rangle_*$  acts on  $e_{n*}$  via the map  $p \mapsto 0$  and  $v_i \mapsto u^{w(i)} u_i$  for  $1 \leq i \leq n$ . Thus we need to compute the homology of

$$\mathcal{K} \otimes_{BP\langle n \rangle_*} e_{n*} = K(0, u_i u^{w(i)} : 1 \leq i \leq n) =: \tilde{\mathcal{K}}.$$

where we use our convention that  $u_n = 1$ . As in the proof of Lemma 3.1, it is clear that the cycle  $\bar{p}$  splits of and yields a tensor factor  $E_{\mathbb{F}_p}(\bar{p})$ . So we only need to consider the Koszul complex on the remaining elements. Let  $\bar{v}_i \in \tilde{\mathcal{K}}$  be the generator of  $\tilde{\mathcal{K}}$  with  $\partial \bar{v}_i = u_i u^{w(i)}$  for  $1 \leq i \leq n$ .

Let  $A$  be the exterior algebra over  $e_{n*}$  as above. Consider the map

$$\begin{aligned} A &\xrightarrow{\psi} \tilde{\mathcal{K}} \\ f'_I &\mapsto \frac{1}{\alpha_I} \partial(\bar{v}_I) \end{aligned}$$

where  $\alpha_I := u^{w(i_0)}$  with  $i_0 = \min(I)$ . We are going to show that this is

- (1) a well-defined map of algebras,
- (2) the image of  $\psi$  equals the subalgebra of cycles of  $\tilde{\mathcal{K}}$ , so it induces a surjection  $\psi' : A \rightarrow H_*(\tilde{\mathcal{K}})$ ,  
and
- (3) the kernel of  $\psi'$  equals the ideal  $\mathfrak{a}$  from above.

This clearly implies our claim. Before we prove the claimed properties of  $\psi$  we need to set up some notation. We consider the lexicographic order on the power set of  $[n]$ . For any element  $c = \sum_{J \subseteq [n]} c_J \bar{v}_J \in \tilde{\mathcal{K}}$  we call  $\text{supp}(c) := \{J \subseteq [n] \mid c_J \neq 0\}$  its support. Moreover, the *leading term* of  $c$  is  $c_J \bar{v}_J$  for  $J = \max(\text{supp}(c))$ .

(1) To see that  $\psi$  is well-defined, consider a set  $I \subseteq [n]$  and set  $i := \min I$ . The map  $w$  is monotonic, and hence it holds that  $u^{w(i)} \mid \partial \bar{v}_k$  for all  $k \in I$ , hence  $\psi$  is well-defined. We write  $f''_I := \psi(f'_I)$  for  $I \subseteq [n]$ . Note that the leading term of  $f''_I$  equals  $u_i \bar{v}_{I \setminus i}$ .

(2) Next, note that each  $f''_I$  is a cycle, because  $0 = \partial \partial \bar{v}_I = \alpha_I \partial f''_I$  and  $\alpha_I$  is a nonzerodivisor on  $\tilde{\mathcal{K}}$ . We are going to show that every cycle in  $\tilde{\mathcal{K}}$  can be written as a linear combination of some  $f''_I$ . So consider a cycle  $c := \sum_J c_J \bar{v}_J \in \tilde{\mathcal{K}}$ . Let  $J_0 := \max(\text{supp}(c))$  and  $j_0 := \min(J_0)$ . The  $\bar{v}_I$  are linearly independent, so we can consider the coefficient of  $\bar{v}_{J_0 \setminus j_0}$  in  $\partial c$  to obtain that

$$c_{J_0} \partial \bar{v}_{j_0} + \sum_{j \in [n] \setminus J_0} c_{(J_0 \setminus j_0) \cup j} \partial \bar{v}_j = 0$$

By the definition of  $J_0$  and  $j_0$ , only the coefficients  $c_{(J_0 \setminus j_0) \cup j}$  with  $j < j_0$  are nonzero. Hence  $c_{J_0} \partial \bar{v}_{j_0} = c_{J_0} u^{w(j_0)} u_{j_0}$  is contained in the ideal  $(u_j \mid 1 \leq j < j_0) \subset e_{n*}$ . As  $u_{j_0}$  is a non-zerodivisor modulo this ideal or a unit and  $u$  is an indeterminate, it follows that already  $c_{J_0}$  is contained in the



ideal. Hence there is a presentation of  $c_{J_0}$  as  $c_{J_0} = \sum_{1 \leq j < j_0} s_j u_j$  with  $s_j \in e_{n*}$ ,  $1 \leq j < j_0$ . Consider the cycle

$$c_1 := \sum_{1 \leq j < j_0} s_j f''_{J_0 \cup j}.$$

Since  $j < j_0 = \min(J_0)$ , the leading term of  $c_1$  is

$$\sum_{1 \leq j < j_0} s_j u_j \bar{v}_{(J_0 \cup j) \setminus j} = \left( \sum_{1 \leq j < j_0} s_j u_j \right) \bar{v}_{J_0} = c_{J_0} \bar{v}_{J_0}.$$

Hence  $c' := c - c_1$  is a cycle with a strictly smaller leading term than  $c$ . As there are only finitely many sets  $I \subseteq [n]$ , the claim follows by induction.

(3) Let  $\mathfrak{a}' \subset A$  denote the kernel of  $\psi'$ . First, we show that  $\mathfrak{a} \subseteq \mathfrak{a}'$ . It is clear that  $u^{w(i)} u_i \in \mathfrak{a}'$ , because  $\psi(u^{w(i)} u_i) = \partial \bar{v}_i$  for each  $i$ . Similarly,  $u^{w(\min I)} f'_I \in \mathfrak{a}'$  for  $I \subseteq [n]$ , because  $\psi(u^{w(\min I)} f'_I) = \partial \bar{v}_I$ . Next, we show that  $u_a f'_{I \cup b} - u_b f'_{I \cup a} \in \mathfrak{a}'$  for  $I \subseteq [n]$ ,  $a, b \in [n]$  with  $a, b < \min(I)$ . For this, we compute

$$\begin{aligned} 0 &= \partial \partial \bar{v}_{\{a,b\} \cup I} = \partial \left( \partial \bar{v}_a \bar{v}_{b \cup I} - \partial \bar{v}_b a \cup I + \sum_{i \in I} \partial \bar{v}_i \bar{v}_{\{a,b\} \cup I \setminus i} \right) \\ &= \partial \bar{v}_a \partial \bar{v}_{b \cup I} - \partial \bar{v}_b \partial \bar{v}_{a \cup I} + \sum_{i \in I} \partial \bar{v}_i \partial \bar{v}_{\{a,b\} \cup I \setminus i} \\ &= u_a u^{w(a)} u^{w(b)} f''_{b \cup I} - u_b u^{w(b)} u^{w(a)} f''_{a \cup I} + \sum_{i \in I} u_i u^{w(i)} \partial \bar{v}_{\{a,b\} \cup I \setminus i} \\ &= u_a u^{w(a)} u^{w(b)} \left( u_a f''_{b \cup I} - u_b f''_{a \cup I} + \sum_{i \in I} u_i u^{w(i)-w(a)-w(b)} \partial \bar{v}_{\{a,b\} \cup I \setminus i} \right). \end{aligned}$$

An elementary computation shows that  $w(i) - w(a) - w(b) \geq 0$  if  $i > a, b$ . Thus we have that

$$u_a f'_{b \cup I} - u_b f'_{a \cup I} + \sum_{i \in I} u_i u^{w(i)-w(a)-w(b)} \alpha_{\{a,b\} \cup I \setminus i} f'_{\{a,b\} \cup I \setminus i} \in \mathfrak{a}'.$$

We denote this element by  $r_{a,b,I}$ . It follows that  $u_a f'_{I \cup b} - u_b f'_{I \cup a} \in \mathfrak{a}'$ .

Next, we show that the relations involving  $f'_I f'_J$  are contained in  $\mathfrak{a}'$ . For this let  $I, J \subseteq [n]$  with  $i_0 := \min(I), j_0 := \min(J)$ . By symmetry we may assume that  $i_0 \leq j_0$ . We start with a short computation:

$$\begin{aligned} f''_I \cdot f''_J &= \frac{1}{\alpha_I \alpha_J} (\partial \bar{v}_I) (\partial \bar{v}_J) = \frac{1}{\alpha_I \alpha_J} \partial (\bar{v}_I \cdot \partial \bar{v}_J) = \frac{1}{\alpha_J} \sum_{j \in J} \partial (\bar{v}_j) \frac{1}{\alpha_I} \partial (\bar{v}_I \cdot \bar{v}_{J \setminus j}) \\ &= \sum_{j \in J} u_j u^{w(j)-w(j_0)-w(i_0)} \partial (\bar{v}_I \cdot \bar{v}_{J \setminus j}). \end{aligned}$$

For  $j > j_0$  we have that  $w(j) - w(j_0) - w(i_0) \geq 0$ , because  $i_0 \leq j_0$ . Thus, the only term which is possibly nonzero is the one with  $j = j_0$ . If  $I \cap J \setminus j_0 \neq \emptyset$ , then  $\bar{v}_I \cdot \bar{v}_{J \setminus j_0} = 0$ . Otherwise, we have that  $u_{j_0} u^{w(j_0)-w(j_0)-w(i_0)} \partial (\bar{v}_I \cdot \bar{v}_{J \setminus j_0}) = u_{j_0} f''_{I \cup J \setminus j_0}$ . Hence the product of  $f'_I$  and  $f'_J$  is as claimed.

It remains to show that  $\mathfrak{a}' \subseteq \mathfrak{a}$ . In homological degree 0, it is clear that  $\psi(r) = 0$  exactly if  $\psi(r)$  is a boundary, and  $\mathfrak{a}$  contains all the preimages of the boundaries in degree 0. Our next step is to show that there are no other linear relations among the  $f''_I$  besides those in  $\mathfrak{a}$ . This is very similar to our argument above. For this, consider an element  $r := \sum_J c_J f'_J \in \mathfrak{a}'$ . Then  $\psi(r)$  is a boundary, so there

exists an element  $c = \sum_{J'} c'_{J'} \bar{v}_{J'}$  such that  $\partial c = \psi(r)$ . In other words,

$$(1) \quad 0 = \sum_J c_J f''_J - \sum_{J'} c'_{J'} \partial \bar{v}_{J'} = \sum_J c_J f''_J - \sum_{J'} c'_{J'} \alpha_{J'} f''_{J'} = \psi \left( \sum_J c_J f'_J - \sum_{J'} c'_{J'} \alpha_{J'} f'_{J'} \right).$$

As  $\alpha_{J'} f'_{J'} \in \mathfrak{a}$  for all  $J'$ , we may replace  $r$  by  $\sum_J c_J f'_J - \sum_{J'} c'_{J'} \alpha_{J'} f'_{J'}$  and thus assume that  $\psi(r) = 0$ .

Next, let  $J_0$  be the (lexicographically) largest set in the  $r$ , and let  $j_0 := \min(J_0)$ . Considering the coefficient of  $\bar{v}_{J_0 \setminus j_0}$  in  $\psi(r)$ , we see that

$$c_{J_0} u_{j_0} + \sum_{j \in [n] \setminus J_0} c_{(J_0 \setminus j_0) \cup j} u_j = 0.$$

Now we argue as above that  $c_{J_0}$  is contained in the ideal  $(u_j \mid 1 \leq j < j_0) \subset e_{n*}$  and can thus be written as  $c_{J_0} = \sum_{1 \leq j < j_0} s_j u_j$  with  $s_j \in e_{n*}$ ,  $1 \leq j < j_0$ . Consider the element

$$r_1 := \sum_{1 \leq j < j_0} s_j r_{j, j_0, J \setminus j_0} \in \mathfrak{a}',$$

where  $r_{j, j_0, J \setminus j_0}$  is the element defined above. As before,  $r$  and  $r_1$  have the same leading term, so we may replace  $r$  by  $r - r_1$ . By induction, it follows that  $r$  can be written as a sum of terms of the form  $\alpha_I f'_I$  and  $r_{a,b,J}$ , and thus  $r \in \mathfrak{a}$ . □

In addition to its algebra structure,  $\text{Tor}_*^{BP*}(\text{H}_*(BP; \mathbb{F}_p), e_{n*})$  also carries the structure of ternary (and higher) Massey products. See Section 2.4 for definitions and conventions regarding Massey products. The following result about these is crucial for our application. It has been inspired by [BR08, Proposition 5.3].

**Proposition 3.3.** *Let  $I, J \subseteq [n]$  be two disjoint sets. Let  $j_0 := \min(J)$  and assume that  $j_0 > \min(I)$ . Then  $(-1)^{\#I+m(I,J)} f_{I \cup J} \in \langle f_I, u^{w(j_0)}, f_J \rangle$  in  $\text{Tor}_*^{BP*}(\text{H}_*(BP; \mathbb{F}_p), e_{n*})$ .*

Recall the notation  $m(I, J) = \#\{(i, j) \in i \times j \mid i > j\}$ .

*Proof.* Let  $i_0 := \min(I)$ . Recall our convention that  $\hat{r} = (-1)^{|r|+1} r$ , where  $r \in \text{Tor}_*^{BP*}(\text{H}_*(BP; \mathbb{F}_p), e_{n*})$  and  $|r|$  denotes its total degree. However, all elements of our algebra have even internal degree, so we may use the homological degree instead of the total degree. Note that  $\hat{f}_I u^{w(j_0)} = (-1)^{\#I} \partial(u^{w(j_0)-w(i_0)} \bar{v}_I)$  and  $\hat{u}^{w(j_0)} f_J = (-1)^{0+1} \partial \bar{v}_J$ . Hence the class of the element

$$\begin{aligned} & ((-1)^{\#I} u^{w(j_0)-w(i_0)} \bar{v}_I) \cdot f_J + (-1)^{\#I+1} f_I \cdot \bar{v}_J = -u^{w(j_0)-w(i_0)} \bar{v}_I \cdot f_J + (-1)^{\#I+1} f_I \cdot \bar{v}_J \\ & = -\bar{v}_I \cdot \frac{1}{\alpha_I} \partial(\bar{v}_J) + (-1)^{\#I+1} \frac{1}{\alpha_I} \partial(\bar{v}_I) \cdot \bar{v}_J \\ & = (-1)^{\#I} \frac{1}{\alpha_I} \left( \partial(\bar{v}_I) \cdot \bar{v}_J + (-1)^{\#I} \bar{v}_I \cdot \partial(\bar{v}_J) \right) \\ & = (-1)^{\#I} \frac{1}{\alpha_I} \partial(\bar{v}_I \bar{v}_J) \\ & = (-1)^{\#I+m(I,J)} f_{I \cup J} \end{aligned}$$

is contained in the Massey product  $\langle f_I, u^{w(j_0)}, f_J \rangle$ . □

**3.2. The collapse of the spectral sequence.** We will also use a result due to Kochman [Koc96] regarding Massey products to conclude from our computations above that the spectral sequence collapses. In this setting, it was established by Baker and Richter in [BR08], see in particular the proof of Theorem 7.3 and Appendix B. In [BR08] they work with commutative ring spectra, but this is not necessary. All that is required for Theorem 2.15 is that the spectral sequence be multiplicative.

We are now able to prove our main theorem.

**Theorem 3.4.** *The spectral sequence*

$$\mathrm{Tor}_s^{BP_*}(\mathrm{H}_*(BP; \mathbb{F}_p), e_{n*})_t \Rightarrow \mathrm{H}_{s+t}(e_n; \mathbb{F}_p)$$

*collapses at the  $E_2$ -page when  $n \in \{1, 2, 3, 4\}$ .*

We now apply the technique used by Baker and Richter in their proof of Theorem 7.3 of [BR08]. Our argument follows theirs. The case when  $n = 1$  is classical as  $e_1 = ku_p$ , the  $p$ -completion of the connective complex  $K$ -theory spectrum.

*Proof.* There are 4 different types of elements in our  $E_2$ -page. There are elements that are contributed by  $\mathrm{H}_*(BP; \mathbb{F}_p)$  and  $e_{n*}$  that are on the 0-line and are necessarily permanent cycles. There are the classes  $\bar{v}_I$  where all elements of  $I$  are larger than  $n$ . These are products of  $\bar{v}_i$  for  $i > n$  and so they are also permanent cycles by the multiplicativity of the spectral sequence. Finally, we have the classes  $f_I$ , and we must show that these are permanent cycles. This will establish the collapse of the spectral sequence at the  $E_2$ -page.

The proof follows by induction on total degree. The result will follow if we are able to show that every  $f_I$  is a permanent cycle since these form a basis for the  $E_2$ -term. Firstly,  $E_2 = E_3$  as  $d_2$  increases the internal degree by 1 and every element in the Tor group has even internal degree. First, it is clear that the  $f_{i,j}$  are permanent cycles as they are on the 1-line of the spectral sequence and all differentials they could support decrease Tor-degree by at least 2. Further, the relation  $\alpha_{i,j} f_{i,j} = 0$  persists through Tor to  $\mathrm{H}_*(e_n; \mathbb{F}_p)$ . This is because there are no elements of odd total degree on the 0-line so there is no room for a multiplicative extension. Next we see that the  $f_{i,j,k}$  are permanent cycles. The only possible nontrivial differentials on them, as they lie on the 2-line of the spectral sequence, are  $d_2$ 's but these are all 0.

Now from this we can deduce that  $f_I$  where  $I = \{i_1, i_2, i_3, i_4\}$  is a permanent cycle. By Proposition 3.3 we have that  $f_I \in \langle f_{i_1, i_2}, u^{p^{i_3}-1}, f_{i_3, i_4} \rangle$ . The indeterminacy of this Massey product consists of permanent cycles as they are decomposable with respect to the product structure.

The classes  $f_{i_1, i_2}$ ,  $u^{p^{i_3}-1}$ , and  $f_{i_3, i_4}$  are each permanent cycles which detect homotopy classes. Since we have that  $\alpha_{i,j} f_{i,j} = 0 \in \mathrm{H}_*(e_n; \mathbb{F}_p)$  we can form the Toda bracket of these homotopy classes. By Theorem 2.15, we have that the element  $f_I$  in the  $E_2$ -page detects an element in the Toda bracket  $\langle f_{i_1, i_2}, u^{p^{i_3}-1}, f_{i_3, i_4} \rangle$  as long as there are no nonzero crossing differentials. This is indeed the case as the domains of the possible crossing differentials are in lower total degree than  $n$  and so must be trivial. Thus  $f_I$  detects an element in  $\mathrm{H}_*(e_n; \mathbb{F}_p)$  as desired.  $\square$

The above argument shows that in fact  $f_I$  detects an element in  $\mathrm{H}_*(e_n; \mathbb{F}_p)$  for all  $n$  when  $I$  has cardinality no larger than 4. It seems unlikely that this approach can be pushed further as we as of yet have no way of showing that the product  $\alpha_I f_I$  is not divisible by an element of  $\mathrm{H}_*(BP; \mathbb{F}_p)$  in general. However, we are able to say something about the closely related spectrum  $\pi_*(\mathrm{HF}_p \wedge_{BP} e_5)$ .

**Proposition 3.5.** *The spectral sequence*

$$\mathrm{Tor}_s^{BP_*}(\mathbb{F}_p, e_{5*})_t \Rightarrow \pi_{s+t}(\mathrm{HF}_p \wedge_{BP} e_5)$$

*collapses at the  $E_2$ -page.*

*Proof.* The argument above works to establish that everything is a permanent cycle with the exception of the element  $f_I$  where  $I = \{1, 2, 3, 4, 5\}$ . This will follow from the fact that  $u^{p^3-1}f_{3,4,5} = 0$  in  $\pi_*(\mathrm{HF}_p \wedge_{BP} e_5)$ . The bidegree of  $u^{p^3-1}f_{3,4,5}$  is  $(2, 2p^3 - 2 + 2p^4 - 2 + 2p^5 - 2)$  and it has total degree  $2p^3 + 2p^4 + 2p^5 - 4$ . This product could be nonzero in the target if there were an element in lower filtration and the same total degree. The only elements in filtration and positive total degree are multiples of  $u$ . In order to reach that total degree we would need to have the product divisible by  $p^5$  at least, but  $u^{p^5-1} = 0$  already. In filtration 1 we have  $f_{i,j}$  and these are all of odd degree so no product of them and a power of  $u$  could have the right total degree. Therefore we have that  $u^{p^3-1}f_{3,4,5}$  is 0 in the target of the spectral sequence and not just the associated graded. Now we use Proposition 3.3 and Theorem 2.15 to show that  $f_I$  is a permanent cycle.  $\square$

While this is not a direct computation regarding the homology of  $e_5$ , it does give us a lot of information since the  $E_2$ -page splits as a tensor product by Lemma 3.1. This relative smash product  $\pi_*(\mathrm{HF}_p \wedge_{BP} e_5)$  still contains all of the new interesting classes  $f_I$ .

Note that these results also imply that the spectral sequence

$$\mathrm{Tor}_s^{BP\langle n \rangle*}(\mathbb{F}_p, e_{n*})_t \Rightarrow \pi_{s+t}(\mathrm{HF}_p \wedge_{BP\langle n \rangle} e_n)$$

collapses. This follows as the map of associative  $S$ -algebras

$$BP \longrightarrow BP\langle n \rangle$$

induces a map of spectral sequences and so we can compute the differentials on all classes in the target by computing them in the source since the map is surjective on  $\mathrm{Tor}$ . This fact will be used in the next section.

**3.3. Multiplicative extensions.** In this section we show that many relations of the form  $xy = 0$  in the  $E_\infty$ -page of the spectral sequence

$$\mathrm{Tor}_s^{BP\langle n \rangle*}(\mathbb{F}_p, e_{n*})_t \Rightarrow \pi_{s+t}(\mathrm{HF}_p \wedge_{BP\langle n \rangle} e_n)$$

in fact hold in homotopy as well. After this we establish that

$$H_*(e_n; \mathbb{F}_p) \cong H_*(BP\langle n \rangle; \mathbb{F}_{p^n}) \otimes_{\mathbb{F}_{p^n}} \pi_*(\mathrm{HF}_p \wedge_{BP\langle n \rangle} e_n).$$

We will use the collapse of the above spectral sequences established in Theorem 3.4, therefore  $n \leq 4$  throughout this section.

**Proposition 3.6.** *We have the following relations in the ring  $\pi_*(\mathrm{HF}_p \wedge_{BP\langle n \rangle} e_n)$ .*

- $u_i u^{p^i-1} = 0$  and  $u^{p^n-1} = 0$ ,
- $u^{p^i-1} f_{i,j} = 0$ ,
- $\alpha_I f_I = 0$  and  $f_I^2 = 0$  whenever  $n \in I$ ,
- when  $p = 2$  we have that  $f_{i,j}^2 = 0$ ,
- $f_{1,2,3}^2 = 0$ .

Since we are working with graded commutative rings, squares of odd degree elements are always 0, except when  $p = 2$ . The only relation of the form  $xy = 0$  that these do not cover is  $u^{p-1}f_{1,2,3} = 0$  when  $n = 4$ .

*Proof.* The relations regarding  $u$  and  $u_i$  hold because they take place in filtration 0 and so there is no room for possible extensions. Recall from the proof of Theorem 3.4 that  $u^{p^i-1}f_{i,j} = 0$  as it is in odd total degree and the only elements in lower filtration are in even total degree. This is also our base case for the induction proof of the next relation.

In each of the following cases, all we have to do is show that there are no eligible candidates in the given total degree of filtration less than that which the relation occurs in. Since we have a basis for Tor and hence  $\pi_*(\mathbb{H}\mathbb{F}_p \wedge_{BP\langle n \rangle} e_n)$ , this amounts to ruling out classes of the form  $qu^k f_I$  where  $q \in \pi_0(e_n)$  and  $k$  and  $I$  are of the appropriate degree. Sometimes  $q$  will not play a role as  $u^k f_I$  is already 0.

First let us consider  $\alpha_I f_I = 0$  when  $n \in I = \{i_1, i_2, i_3, \dots, i_m\}$ . Assume that this is true for all  $J$  of cardinality less than  $m$ . Thus we are looking for an element in total degree  $(-1) + \sum_{i \in I} 2p^i - 1$  in filtration less than  $m - 1$ . This is impossible for degree reasons. Since the parity of the total degree is the same as the parity of  $\#I - 1$  we see that the only way to have an element in the same total degree is to be in an even number of filtrations lower. So the next element in a lower filtration of the highest possible total degree is  $qu^j f_{I''}$  where  $I''$  is  $I$  without it's two smallest elements for some  $j$ . The difference in total degree between  $f_{I''}$  and  $\alpha_I f_I$  is  $2p^{i_1} - 1 + 2p^{i_2} - 1 + 2p^{i_3} - 1$  thus  $j > p^{i_3} - 1$ . However, this product is already 0 by our hypothesis. This relies on the already observed fact that  $u^{p^{i_3}-1} f_{i,j} = 0$ .

The relation  $f_I^2$  is more straightforward. Suppose that  $qu^k f_J$  is in the same total degree as  $f_I^2$ , which is  $(-2) + \sum_{i \in I} 4p^i - 2$ . By the relation  $\alpha_J f_J = 0$  we see that  $k < p^{j_1} - 1$ . Therefore the total degree of  $qu^k f_J$  is  $\sum_{j \in J} (2p^j - 1) - 1$ . However this will never be large enough as the total degree of  $f_I^2$  is larger than  $4p^n - 2$  and we have

$$\begin{aligned} 4p^n - 2 &> \sum_{i=1}^n 2p^i - 1 \\ &> \sum_{j \in J} 2p^j - 1 \\ &> |qu^k f_J|. \end{aligned}$$

Next we consider the class  $f_{i,j}$  in total degree  $2^{j+1} - 1$ . Its square is in filtration 2 and total degree  $2 \cdot 2^{j+1} - 2$  and so  $qu^{2^{j+1}-1}$  is the only possible class other than 0 that  $f_{i,j}^2$  could be. If  $j = n - 1$  then this power of  $u$  is 0. If  $j \neq n - 1$  then we obtain the relation  $u^{2^i-1} qu^{2^{j+1}-1} = 0$ . But this can not be 0 unless  $q$  is divisible by  $u_k$  for  $k < i$ . If this were the case though then  $qu^{2^{j+1}-1} = 0$  since  $i < j$ .

The last case is the square of the element  $f_{1,2,3}$  in total degree  $2p^2 - 1 + 2p^3 - 1$ . This is covered by other cases except when  $n = 4$ . The possible elements in the same total degree are  $a := qu^{2p^2-1+2p^3-1}$  and  $qu^m f_{i,j,k}$  in total degree  $2m + 2p^j - 1 + 2p^k - 1$  for  $m < p^i - 1$ . We will deal with these two cases separately.

First let us consider the case  $a$ . Note that  $q = 1$  since if it were divisible by  $u_i$  then  $a = 0$  since  $u_i u^{p^i-1} = 0$ . At the prime 2 the element  $a = 0$ . Otherwise we have that  $u^p - 1a = 0$ . This contradicts the fact that  $u^{p^4-2} \neq 0$  when  $p > 2$ .

Now consider the possibility that  $f_{1,2,3}^2 = qu^m f_{i,j,k}$ . This element is annihilated by  $u^{p-1}$  since  $f_{1,2,3}$  is. If  $k \neq 4$  then  $f_{i,j,k}$  must be  $f_{1,2,3}$  and  $m < p - 1$  so the element  $qu^m f_{i,j,k}$  will not be in high enough total degree. However, the element  $f_{i,j,4}$  has higher total degree than  $f_{1,2,3}^2$  unless  $p = 2$ . When  $p = 2$ ,  $|f_{1,2,3}^2| = 44$  and the only element in this total degree is  $qu^3 f_{1,2,4}$  which is 0 by the above relation  $\alpha_I f_I = 0$  when  $n \in I$ .

□

Now we establish that  $H_*(e_n; \mathbb{F}_p)$  splits as a tensor product. The following ring maps will help us split the homology of connective Morava  $E$ -theory.

$$\pi_*(\mathbb{H}\mathbb{F}_p \wedge_{BP\langle n \rangle} e_n) \xleftarrow{\psi} H_*(e_n; \mathbb{F}_p) \xrightarrow{\varphi} H_*(BP\langle n \rangle; \mathbb{F}_{p^n})$$

The ring map  $\psi$  is induced by the maps

$$S \longrightarrow BP \longrightarrow BP\langle n \rangle,$$

where  $S$  is the sphere spectrum, and is therefore a map of rings. The map  $\varphi$  is constructed by first mapping to the dual Steenrod algebra tensored up to  $\mathbb{F}_{p^n}$  and then seeing that the image of  $H_*(e_n; \mathbb{F}_p)$  is contained in  $H_*(BP\langle n \rangle; \mathbb{F}_{p^n})$ . To compute each map involved we will consider the relevant map of spectral sequences where the  $E_2$ -pages can always be computed using the “same” underlying Koszul complex. Here we record some basic facts about the above maps.

**Proposition 3.7.**

- The map  $\varphi$  takes each  $u, u_i, f_I \in H_*(e_n; \mathbb{F}_p)$  to 0.
- The map  $\psi$  takes all classes coming from  $H_*(BP; \mathbb{F}_p)$  as well the  $\bar{v}_I$  to 0.
- The classes  $\bar{v}_{n+k} \in \text{Tor}_1^{BP*}(H_*(BP; \mathbb{F}_p), e_{n*})_*$  are sent by  $\varphi$  to the conjugates of the classes  $\xi_{n+k+1}$  or  $\tau_{n+k}$  in the dual Steenrod algebra, when the prime is 2 or odd respectively.
- The map  $\varphi$  factors as

$$\begin{array}{ccc} H_*(e_n; \mathbb{F}_p) & \xrightarrow{\varphi} & H_*(H\mathbb{F}_p; \mathbb{F}_{p^n}) \\ & \searrow & \nearrow \\ & H_*(BP\langle n \rangle; \mathbb{F}_{p^n}) & \end{array}$$

*Proof.* To see each of these we look at the induced maps on the Koszul complexes. Since  $f_{i,j}$  involves classes in  $e_{n*}$  that are sent to 0 in  $\mathbb{F}_p$  we have that  $\varphi(f_{i,j}) = 0$  since there is nothing in lower filtration and the same internal degree. We also have that  $\varphi(u) = 0 = \varphi(u_i)$ . Since the map  $\varphi$  is induced by a map of commutative ring spectra, it takes Toda brackets to Toda brackets. Thus  $\varphi(f_I) \in \varphi(\langle f_{i,j}, \alpha_{I'}, f_{I'} \rangle) \subset \langle 0, 0, 0 \rangle = \{0\}$  where  $I' = I \setminus \{i, j\}$ .

It is obvious that  $\psi$  does this by considering the following map of spectral sequences

$$\text{Tor}^{BP*}(H_*(BP; \mathbb{F}_p), e_{n*}) \longrightarrow \text{Tor}^{BP*}(\mathbb{F}_p, e_{n*}) \longrightarrow \text{Tor}^{BP\langle n \rangle*}(\mathbb{F}_p, e_{n*})$$

each of which comes from a map of ring spectra. All of the above spectral sequences collapse since the first one does. Then notice that there are no elements in lower filtration for the  $\bar{v}_{n+k}$  to be sent to.

We establish the third claim by considering the map of spectral sequences induced by

$$e_n \longrightarrow H\mathbb{F}_p.$$

We use the same Koszul complex to compute  $\text{Tor}$ . Each class  $f_I$  is taken to zero. We also understand the spectral sequence

$$\text{Tor}_s^{BP*}(H_*(BP; \mathbb{F}_p), \mathbb{F}_p) \Rightarrow H_*(H\mathbb{F}_p; \mathbb{F}_p)$$

completely as we know what it converges to and so it must collapse. That these classes detect the conjugates follows from discussion in Chapter 4 Section 2 [Rav86].

The rest follows by considering the map of spectral sequences

$$\text{Tor}_*^{BP*}(H_*(BP; \mathbb{F}_p), e_{n*})_* \longrightarrow \text{Tor}_*^{BP*}(H_*(BP; \mathbb{F}_p), \mathbb{F}_{p^n})_*$$

restricted to the 0-line. This map of rings is induced by the map of commutative  $S$ -algebras

$$e_n \longrightarrow H\mathbb{F}_{p^n}$$

and so induces a map of rings on the 0-line. □

**Lemma 3.8.** *The homology  $H_*(e_n; \mathbb{F}_p)$  splits as a tensor product of rings*

$$H_*(e_n; \mathbb{F}_p) \cong H_*(BP\langle n \rangle; \mathbb{F}_{p^n}) \otimes_{\mathbb{F}_{p^n}} \pi_*(H\mathbb{F}_p \wedge_{BP\langle n \rangle} e_n).$$

*Proof.* We have the two maps of rings  $\varphi$  and  $\psi$ . They induce a map of rings

$$H_*(e_n; \mathbb{F}_p) \xrightarrow{\varphi \times \psi} H_*(BP\langle n \rangle; \mathbb{F}_{p^n}) \times \pi_*(H\mathbb{F}_p \wedge_{BP\langle n \rangle} e_n).$$

This composed with the canonical map

$$H_*(BP\langle n \rangle; \mathbb{F}_{p^n}) \times \pi_*(H\mathbb{F}_p \wedge_{BP\langle n \rangle} e_n) \longrightarrow H_*(BP\langle n \rangle; \mathbb{F}_{p^n}) \otimes \pi_*(H\mathbb{F}_p \wedge_{BP\langle n \rangle} e_n)$$

gives us the desired splitting. To see that it is an isomorphism we note that it is injective and surjective as a map of  $F_p$ -modules since the spectral sequence collapses.  $\square$

Note that this is a splitting of rings as there not known map of spectra from

$$H\mathbb{F}_p \wedge e_n \longrightarrow H\mathbb{F}_p \wedge BP\langle n \rangle$$

or from

$$H\mathbb{F}_p \wedge_{BP\langle n \rangle} e_n \longrightarrow H\mathbb{F}_p \wedge e_n.$$

This resolves the issue of there being multiplicative extensions where products of elements coming from  $\pi_* H\mathbb{F}_p \wedge_{BP\langle n \rangle} e_n$  become divisible by elements from  $H_*(BP\langle n \rangle; \mathbb{F}_{p^n})$  and vice versa.

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